Numerical solution of non-linear Volterra integral equations

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ABSTRACT

This paper deals with non-linear Volterra integral equations of the type
$y(x) = f(x) + \int_0^x H[t, x, y(t), y(x)]dt$.

Convergence criteria are given (in the same sense of the maximum and C^α norms) for the numerical solution of this type of Volterra integral equation. Several numerical methods are compared.

INTRODUCTION

In this paper we study both theoretical and numerical aspects of solutions of non-linear Volterra integral equations of the form

$$y(x) = f(x) + \int_0^x H[t, x, y(t), y(x)]dt, \qquad 0 \leqslant x \leqslant a . \tag{0.1}$$

Problems of this form arise in the theory of radiative transfer [1], and in astrophysics [2].

Numerical methods for solving non-linear Volterra integral equations of the form

$$y(x) = f(x) + \int_0^x H[t, x, y(t)]dt, \qquad 0 \leqslant x \leqslant a \tag{0.2}$$

were considered, e.g., in [3, 4, 5, 6]. In these papers convergence criteria are only given in the sense of maximum norm.

In the present paper we consider a Volterra integral equation of the form (0.1). We establish existence and uniqueness of the solution of equation (0.1) that belongs to the space $C^\alpha[0, a]$ for a sufficiently small $(0 \leqslant \alpha < 1)$. The space $C^\alpha[0, a]$ is the Banach space of all continuous functions g with the norm

$$\|g\|_{C^\alpha} = \max_{0 \leqslant x \leqslant a} |g(x)| + \sup_{x_1, x_2 \in [0,a]} \frac{|g(x_1) - g(x_2)|}{|x_1 - x_2|^\alpha} < \infty, \qquad (0 \leqslant \alpha < 1).$$

For the numerical solution we consider equation (0.1) separately in two cases depending on form of $H[t, x, y(t), y(x)]$. In the first case we consider $H[t, x, y(t), y(x)]$ to be continuous with respect to t and to have bounded second derivatives with respect to the second, third and fourth arguments. General criteria to decide when various different numerical schemes, not depending on x give solutions converging to the solution of equation (0.1) in the sense of the C^α norm are given. (Definition of this type of convergence is given in equation (1.10). In the second case we assume that

$H[t, x, y(t), y(x)] = k(t, x)h[t, x, y(t), y(x)]$,

where for each x, k (t, x) is an integrable function, and where $h[t, x, y(t), y(x)]$ satisfies Lipschitz conditions in the third and fourth arguments and is continuous with respect to x and t. Under these assumptions we give quite general criteria to decide when various different numerical schemes depending on x, give solutions converging to the solution of (0.1) in the sense of the maximum norm.

To avoid the complexity and emphasize the results and the methods, we refer the reader to the technical report [7] for the proofs of some of the results.

1. REGULAR KERNELS

In this section, we consider equation (0.1) namely

$$y(x) = f(x) + \int_0^x H[t, x, y(t), y(x)]dt, \qquad 0 \leqslant x \leqslant a \tag{1.1}$$

where

H is continuous with respect to t and has bounded second derivatives with respect to the second, third, and fourth arguments.

In this section, we establish the following results : existence and uniqueness, numerical methods and convergence of the numerical solution.

(i) *EXISTENCE AND UNIQUENESS OF THE SOLUTION*

Theorem 1.1

Let $f(x) \in C^\alpha$ $(0 \leqslant \alpha < 1)$ and assume that $H(t, x, u, v)$ satisfies the following conditions :

(H1) $H(t, x, u, v)$ has bounded second derivatives with respect to x, u and v.

(H2) $H(t, x, u, v)$ is a continuous function with respect to t for $0 \leqslant t \leqslant x$.

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Then there exists a unique solution of equation (0.1) belonging to C^{α}.
For the proof of theorem 1.1, we need the following lemmas.

Lemma 1.1

Let $G(u, v)$ have a bounded second derivative $\frac{\partial^2 G}{\partial u \partial v}$.
Then

$$|G(u_2,v_2)-G(u_2,v_1)-G(u_1,v_2)+G(u_1,v_1)|$$
$$\leq K|u_2-u_1|\,|v_2-v_1| \tag{1.2}$$

where

$$K = \sup \left|\frac{\partial^2 G}{\partial u \partial v}\right|.$$

For the proof see, e.g. [8, p. 220].

Lemma 1.2

Let $y_1(x) . y_2(x)$ belong to C^{α} and suppose that $F(v)$ has a bounded second derivative. Then there exists a function $h(x) \in C^{\alpha}$ (h depending on y_1, y_2) such that

$$F[y_2(x)] - F[y_1(x)] = h(x)\,[y_2(x) - y_1(x)].$$

Moreover we have

$$\frac{|h(x_2)-h(x_1)|}{|x_2-x_1|^{\alpha}} \leq \frac{K}{2}\left[\frac{|y_2(x_2)-y_2(x_1)|}{|x_2-x_1|^{\alpha}} + \frac{|y_1(x_2)-y_1(x_1)|}{|x_2-x_1|^{\alpha}}\right] \tag{1.3}$$

where $K = \sup_{v} |F''(v)|$.

For the proof of lemma 1.2 and then of theorem 1.1 see [7].

(ii) *NUMERICAL METHODS*

To approximate the solution of equation (1.1) we replace it with the system

$$S_n(t_{i,n}) = f(t_{i,n}) + \sum_{k=1}^{i} w_{k,n} H[t_{k,n}, t_{i,n}, s_n(t_{k,n}), s_n(t_{i,n})] \tag{1.4}$$

for $i = 1,\ldots,n$.

Here $w_{k,n}$ are n numerical weight constants and $\{t_{i,n}\}_{i=1}^{n}$ are n grid points such that $0 = t_{1,n} < t_{2,n} < \ldots\ldots < t_{n,n} = a$. When high order quadrature methods are used (1.4) can be used for $i \geq i_0$ and starting values can be completed by other means. See [9].
Our method of solving (1.4) is based on an iteration process in each subinterval.
For convenience we will drop the subscript n from t and w throughout the remainder of this paper.

$$s_n^0(t_k) = f(t_k), \quad k = 1,\ldots,n \tag{1.5}$$
$$s_n^{\ell}(t_j) = f(t_j) + \sum_{k=1}^{j-1} w_k H[t_k, t_j, s_n(t_k), s_n^{\ell-1}(t_j)]$$
$$+ w_j H[t_j, t_j, s_n^{\ell-1}(t_j), s_n^{\ell-1}(t_j)] \quad \text{for } \ell = 1, 2, \ldots$$

Then we take

$$s_n(t_j) = \lim_{\ell \to \infty} s_n^{\ell}(t_j). \tag{1.6}$$

Now we describe a specific method for solving equation (1.1). Divide the interval $[0, a]$ into n equal subintervals.
Let $h = a/(n-1)$ and let $t_i = (i-1)h$, $i = 1,\ldots,n$, and denote $y(t_i)$ by y_i.

Method R1

We define $R_1(t_1) = 0$, and when $i > 1$ is odd, we let $R_1(t_i)$ be the piecewise quadratic polynomial interpolation operator at the points

$t_{2k-1}, t_{2k}, t_{2k+1}, 2k-1 = 1,\ldots, i-2.$

When i is even, we let $R_1(t_i) = R_1(t_{i-1})$ for the points $t_1,\ldots,t_{i-1}$ and at the points t_{i-1}, t_i $R_1(t_i)$ y will be the quadratic polynomial interpolation at the points t_{i-1}, t_i and t_{i+1} whose values at those points are y_{i-1}, y_i and $f(t_{i+1})$. Then replace (1.1) by

$$y_i = f(t_i) + \int_0^{t_i} R_1(t_i) H[t, t_i, y(t), y(t_i)]dt, \quad i = 1,\ldots,n$$

perform the indicated integrations and obtain an equivalent equation of the form (1.4) which is then solved by the iteration scheme of (1.5).

(iii) *CONVERGENCE OF THE NUMERICAL SOLUTION*

Consider the system (1.4). One can show

Corollary 1.1

Assume that H1 - H2 are satisfied and n sufficiently large. Then for a sufficiently small there exists a unique solution to the system (1.4).
We say that the solution of the system (1.4) converges to the solution, $s(x)$, of equation (1.4) on $[0, a]$ with C^{α} norm if

$$\lim_{n\to\infty} \|s_n - \Delta_n s\|_{C^{\alpha}} = \lim_{n\to\infty} \max_{1\leq i\leq n} |s_n(t_i) - s(t_i)|$$
$$+ \lim_{n\to\infty} \max_{t_i, t_j} \frac{|s_n(t_i) - s_n(t_j) - [s(t_i) - s(t_j)]|}{|t_i - t_j|^{\alpha}} = 0 \tag{1.10}$$

where Δ_n is a grid-restriction operator, i.e.

$\Delta_n : C^{\alpha}[0, a] \to R^n$.

It is clear from the definition that convergence in C^{α}

guarantees convergence in C. Moreover, if the rate of convergence is $O(h^{\beta})$, i.e., $\max_{t_i} |s_n(t_i) - s(t_i)| = \check{O}(h^{\beta})$, then the rate of convergence in the sense of the C^{α} norm is $O(h^{\beta-\alpha})$. Therefore, we have more rapid convergence, if $\lim_{n\to\infty} \|s_n - \Delta_n s\|_{C^{\alpha}} = 0$.

Theorem 1.2

Suppose that H1 - H2 are satisfied. Then

$$\lim_{n\to\infty} \|s_n - \Delta_n s\|_{C^{\alpha}} = 0. \tag{1.11}$$

The proof of this theorem appears in [7].

2. SINGULAR KERNELS

We assume in this section that

$H[t, x, y(t), y(x)] = k(t,x)\, h[t, x, y(t), y(x)]$

so that (0.1) takes the form

$$y(x) = f(x) + \int_0^x k(t,x) h[t,x,y(t),y(x)]\,dt \text{ for } 0 \leqslant t \leqslant x \leqslant a, \tag{2.1}$$

where $f(x) \in C[0, a]$ and $k(t, x)$ satisfies the conditions

A1 : For x fixed, $0 \leqslant t \leqslant x \leqslant a$, $k(t,x) \in L^1[0, a]$.

A2 : For x fixed, $0 \leqslant x^1 \leqslant a$,

$$\lim_{x^1 \to x} \int_0^x |\sigma(t,x^1)k(t,x^1) - \sigma(t,x)k(t,x)|\,dt = 0$$

where

$$\sigma(t,x) = \begin{cases} 1 & \text{if } t \leqslant x \\ 0 & \text{if } t > x. \end{cases}$$

Condition A2 means that $k(t, x)$ as a function in $[0, a] \times [0, a]$ is L^1 continuous with respect to x.

In this section, we also establish existence and uniqueness, numerical methods and convergence of the numerical method.

(i) *EXISTENCE AND UNIQUENESS*

Theorem 2.1

Suppose that A1 - A2 are satisfied and $h(t, x, u, v)$ satisfies A3. $h(t, x, u, v)$ is a continuous function with respect to each argument

A4. $|h(t,x,u_1, v_1) - h(t,x,u_2,v_2)| \leqslant L_1 |u_1 - u_2| + L_2 |v_1 - v_2|$

for each $x \in [0, a]$ and $t \leqslant x$.

We also assume that

$$(L_1 + L_2) \int_0^x |k(t,x)|\,dt < 1 - \beta, \text{ for } 0 < \beta < 1 \tag{2.2}$$

and for all $x \in [0, a]$. Then there exists a unique solution to equation (2.1) continuous in $[0, a]$.

Applying the Banach fixed point theorem one can prove theorem 2.1. (See e.g., [10] Chapter 2).

It is clear from A1 that condition (2.2) is satisfied for small a.

(ii) *NUMERICAL METHODS*

In order to replace (2.1) by a numerical scheme, we follow [11] and consider the following.

Define

$$k^+(t,x) = \begin{cases} k(t,x) & k(t,x) \geqslant 0 \\ 0 & k(t,x) < 0 \end{cases}$$

$$k^-(t,x) = \begin{cases} -k(t,x) & -k(t,x) \geqslant 0 \\ 0 & -k(t,x) < 0 \end{cases}.$$

Now suppose that for each integer n we can choose n grid points $0 = t_1 < t_2 < \dots < t_n = a$ and 2n numerical weights functions $w_j^+(x), w_j^-(x)$, $j = 1, \dots, n$, such that if $R[0, a]$ is the space of Riemann integrable functions defined on $[0, a]$, then

A5. $w_j^{\pm}(x) \in R[0, a]$

A6. $w_j^{\pm}(x) \geqslant 0$, $w_1^{\pm}(t_1) = 0$ and

$w_j^{\pm}(t_i) = 0$ for $j > i$

and

A7. $\lim_{n\to\infty} \| \int_0^x k^{\pm}(t,x)y(t)\,dt - \sum_{j=1}^n w_j^{\pm}(x) y(t_j)\| = 0$ for $y \in C[0, a]$.

Here $\|\cdot\|$ denotes the supremum norm on $C[0, a]$.

For example, the $w_j^{\pm}(x)$ can be taken as the terms obtained by applying product integration techniques as developed by Atkinson [12].

We now let

$$w_j(x) = w_j^+(x) - w_j^-(x) \tag{2.3}$$

so that

$$\lim_{n\to\infty} \| \int_0^x k(t,x) - \sum_{j=1}^n w_j(x)\, y(t_j)\| = 0 \text{ for } y \in C[0, a].$$

To find approximate solutions of (2.1), we replace (2.1) with the system

$$s_n(t_i) = f(t_i) + \sum_{k=1}^{i} w_k(t_i)\, h[t_k, t_i, s_n(t_k), s_n(t_i)] \tag{2.4}$$

for $i = 1, \dots, n$. As in (1.4) when high order quadrature methods are used (2.4) can be used for $i \geqslant i_0$ and starting values can be computed by other means (see [9]). Our method of solving (2.4) is based on an iteration process in each subinterval.

I.

$s_n^0(t_k) = f(t_k)$, $k = 1, \dots, n$

$$s_n^{\ell}(t_j) = f(t_j) + \sum_{k=1}^{j-1} w_k(t_j) h[t_k, t_j, s_n(t_k), s_n^{\ell-1}(t_j)]$$

$$+ w_j(t_j) h[t_j, t_j, s_n^{\ell-1}(t_j), s_n^{\ell-1}(t_j)] \qquad (2.5)$$

for $\ell = 1, 2, \ldots$

Then we take

$$s_n(t_j) = \lim_{\ell \to \infty} s_n^{\ell}(t_j). \qquad (2.6)$$

II. Predictor-corrector method

We shall take the predictor to be $s_n^0(t_k) = s_n(t_k)$, $k = 1, \ldots, n$. Then insert this predictor into the corrector, i.e., equation (2.5). In this method the predictor is obtained using another quadrature method. Also in this case, singular kernels, the numerical weights are functions.

Now we describe specific methods for singular kernels: Divide the interval into n equal subintervals. Let $\Delta t = a/n$ and let $t_i = (i-1)\Delta t$, $i = 1, \ldots, n+1$ and denote $y(t_i)$ by y_i.

Method S1

Let $S_1(t_i)$ be the piecewise quadratic polynomial interpolation operator at the points $t_{2m-1}, t_{2m}, t_{2m+1}$, $2m-1 = 1, \ldots, i-2$ for i odd. When i is even let $S_1(t_i)$ be linear interpolation at t_1, t_2 and the piecewise quadratic polynomial interpolation operator at the points $t_{2m}, t_{2m+1}, t_{2m+2}$, $2m = 2, \ldots, i-2$.

Then replace (2.1) by

$$y_i = f(t_i) + \int_0^{t_i} k(t, t_i) S_1(t_i) h[t, t_i, y(t), y(t_i)] dt$$

and solve the equation by using method I.

Method S2

If i is odd, let $S_2(t_i) = S_1(t_i)$. When i is even we let $S_2(t_i) = S_2(t_{i-1})$ for the points $t_1, \ldots, t_{i-1}$ and on the points t_{i-1}, t_i it is defined to be linear interpolation. Then solve as above.

Method S3

If i is odd, $S_3(t_i) = S_2(t_i)$. For $i = 2$, $S_3(t_2) = S_1(t_2)$. If $i > 2$ is even, $S_3(t_i) = S_2(t_{i-3})$ at the points $t_1, \ldots, t_{i-3}$ and S_3 is a cubic interpolation through the points $t_{i-3}, t_{i-2}, t_{i-1}, t_i$. Then solve as before.

Each of the previous three methods requires one to compute y_2 using a two point approximation. As it is felt that, in general, it is better to use methods which require three point approximations, Linz [9] suggested using either of the two following improvements.

Method S4

If i is odd, $S_4(t_i) = S_3(t_i)$. If i is even, $S_4(t_i) = S_2(t_i)$ on $t_1, \ldots, t_{i-1}$. Let $t_{i-1/2} = (t_i - t_{i-1})/2$ and if y_{i-1}, y_i, y_{i+1} are the values of $y(t)$ at t_{i-1}, t_i, t_{i+1} respectively we let $y_{i-1/2} = \frac{3}{8} y_{i-1} + \frac{3}{4} y_i - \frac{1}{8} y_{i-1}$. Then on $[t_{i-1}, t_i]$, $S_4(t_i)y$ will be the quadratic polynomial interpolation through the points $t_{i-1}, t_{i-1/2}, t_i$ whose values at those points are $y_{i-1}, y_{i-1/2}, y_i$, respectively. Then replace (2.1) as before and solve.

Method S5

If $i = 2$, $S_5(t_2) = S_4(t_2)$ and for $i \neq 2$, $S_5(t_i) = S_3(t_i)$. Then solve as before.

We also suggest the following different improvements.

Method S6

If i is odd, let $S_6(t_i) = S_1(t_i)$. When i is even we let $S_2(t_i) = S_2(t_{i-1})$ at the points $t_1, \ldots, t_{i-1}$ and at the points $[t_{i-1}, t_i]$ $S_6(t_i)y$ will be quadratic polynomial interpolation through the points t_{i-1}, t_i, t_{i+1} whose values at those points are $y_{i-1}, y_i, f(t_{i+1})$. Then replace (2.1) as before and solve.

Method S7

Using one of the six methods above to solve numerically equation (2.1), the values of the numerical solution will serve as a predictor. Then solve again by another method from the list above where the initial guess is the values of the predictor. Here we use a predictor-corrector method II.

(iii) *CONVERGENCE OF THE NUMERICAL SOLUTION*

We say that the solutions of the system (2.4) converge to the solution, s(x), of (2.1) on [0, a] with the C norm if

$$\lim_{n\to\infty} \|s_n - \Delta_n s\| = \lim_{n\to\infty} \max_{1 < i < n} |s_n(t_{i,n}) - s(t_{i,n})| = 0. \qquad (2.7)$$

Theorem 2.2

Suppose that A1-A7 are satisfied and that condition (2.2) is satisfied. Then $\lim_{n\to\infty} \|s_n - \Delta_n s\|_C = 0.$ (2.8)

We will state several lemmas which will be needed to prove theorem 2.1.

Lemma 2.1

Suppose that k(t, x) satisfies A1-A2. Then for $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that for all $0 \leq a_1 < a_2 \leq x$ with $0 \leq a_1 < a_2 \leq x$ with $0 < a_2 - a_1 < \delta$ we have

$$\int_{a_1}^{a_2} |k(t, x)| dt < \epsilon. \qquad (2.9)$$

The numerical weight functions $w_j(x)$ satisfy an analog of lemma 2.1.

We denote by $[c, d]_n$ or $[c, d)_n$ the sets of integers j such that $t_j \in [c, d]$ or $t_j \in [c, d)$.

Lemma 2.2

Suppose A1-A2 and A5-A7 are satisfied. Then for each $\epsilon > 0$ there exists points $0 = p_1 < \ldots < p_{R(\epsilon)} = a$ and $N = N(\epsilon)$ such that for all $n \geqslant N$

$$\sum_{j \in [p_i, p_{i+1}]_n} |w_j(x)| < \epsilon, \quad i = 1, \ldots, R(\epsilon) - 1, \qquad (2.10)$$

in $x \in [0, a]$.

For the proof of lemmas 2.1, 2.2 and theorem 2.2, see appendix A. From the proof of theorem 2.2, we conclude

Corollary 2.1

The rate of convergence is

$\|s_n - \Delta_n s\| = 0(Q')$.

where Q' is defined in appendix A (4.5). The rate of convergence is the same order as that of the method of integration.

Lemma 2.2 implies

Corollary 2.2

Suppose A1-A2 and A5-A7 are satisfied. Then for each $\epsilon > 0$ there exists an N such that for all $n > N$, $x \in [0, a]$ and $1 \leqslant j \leqslant n$ we have

$$|w_j(x)| < \epsilon \qquad (2.11)$$

One can also verify

Corollary 2.3

Suppose the conditions (A7) and (2.2) are satisfied. Then for each $\epsilon > 0$ there exists an N such that for all $n > N$, $x \in [0, a]$

$$L_2 \sum_{j=1}^{n} |w_j(x)| < 1 - \frac{3}{4}\beta \qquad (2.12)$$

Corollary 2.4

Suppose the conditions A1-A7 are satisfied then for n sufficiently large there exists a unique solution to the system (2.4).

(iv) *NUMERICAL EXAMPLES*

To test the usefulness of the methods above we consider the example

$$y(x) = \sqrt{x} + \frac{1}{4} x^{3/2} (\ell n\, 2 - 1) + \frac{1}{4} \int_0^x \frac{y^2(t) y(x)}{x + t}\, dt \qquad (3.1)$$

which has the exact solution $y(x) = \sqrt{x}$. The equation was solved by each of the eight methods for $h = .1$, $.02$, $.01$, $.005$ and $.00025$. The results of all the methods for $h = .1$, $.00025$ appear in tables 1-2. Moreover, we calculate the value of

$$E = \frac{|y_{i+1} - y_i - (Y_{i+1} - Y_i)|}{\sqrt{h}}$$

TABLE 1. $h = 0.1$ Equation (3.1)

x	Exact value	Method S1	Method S2	Method S3	Method S4	Method S5	Method S6	Method S7	Method R1	E
.2	.447214	.447135	.445169	.440352	.440352	.440352	.447203	.452084	.447176	1.49895×10^{-3}
.4	.632455	.631164	.628877	.630948	.630948	.630928	.630999	.631583	.630978	3.0449×10^{-4}
.8	.894427	.889832	.886975	.888921	.888921	.888904	.891331	.892034	.891320	3.3064×10^{-4}
1.0	1.000000	.992737	.989704	.991374	.991374	.991358	.996195	.996872	.996185	1.58795×10^{-3}

TABLE 2. $h = 0.0025$ Equation (3.1)

x	Exact value	Method S1	Method S2	Method S3	Method S4	Method S5	Method S6	Method S7	Method R1	E
.2	.447214	.447079	.4471366	.4470381	.447075	.447047	.447181	.447180	.447149	3.37×10^{-6}
.4	.632455	.631810	.632314	.631753	.631797	.631797	.632406	.632403	.632357	6.64×10^{-6}
.8	.894427	.890840	.893889	.890768	.890805	.890805	.894349	.894340	.894271	1.365×10^{-5}
1.0	1.000000	.993686	.998983	.993608	.993638	.993380	.999908	.999893	.999908	1.9287×10^{-4}

In methods S7, we use method S4 as a predictor and in method S5 as a corrector. We find that this choice of method S7 is the best possible choice for method S7.

for method R1 where y_i is the exact value of the solution of (3.1) at the point t_i and Y_i is the approximate value of y_i.
These and all other computations were carried out in single precision arithmetic on the CDC CYBER-73 computer of the Ben Gurion University. All iterations were stopped when the difference between two successive iterations was less than 10^{-9}.
Then we consider the equation

$$y(x) = x - \beta\left(\frac{2}{3} x^2\right) + \beta \int_0^x y(t)y(x)dt, \quad 0<x<1. \tag{3.2}$$

which has the exact solution $y = \sqrt{x}$. It is clear that H satisfies conditions H1-H2 and $f(x) \in C^{1/2}$. Therefore, from theorem 1.1 there exists a unique solution belonging to $C^{1/2}[0, a]$ for a small enough. By direct calculation, one can show from theorem 1.1 that for $0 \leqslant a \leqslant 0.5$ there exists a unique solution belonging to $C^{1/2}[0, a]$ for $a = 1$. Moreover, from theorem 1.2 there is convergence numerically in the sense of $C^{1/2}$ and rapid convergence is expected.

TABLE 3. Errors ($|(y(1) - Y(1)|$) for equation (3.2) at $x = 1$ ($y(1) = 1$) and $\beta = 0.1$

Method	h = 0.1	h = 0.01
S1	4.9951×10^{-4}	4.5718×10^{-4}
S2	4.4917×10^{-4}	0.8686×10^{-4}
S3	7.5045×10^{-4}	4.8197×10^{-4}
S4	3.3806×10^{-4}	3.4522×10^{-4}
S5	3.37798×10^{-4}	3.4522×10^{-4}
S7	0.6808×10^{-4}	1.1154×10^{-4}
R1 (and S6)	2.0501×10^{-4}	0.1864×10^{-4}

We also considered

$$y(x) = x^{0.01} - 25x^{2.02} + 0.25 \int_0^x y(t)y(x)dt, \tag{3.3}$$
$$0 \leqslant x \leqslant 1,$$

which has a unique solution in C, $y(x) = x^{0.01}$, and $y(x) \in C^{\alpha}$ for each $.01 < \alpha < 1$. Therefore, in this case we have a numerical convergence in C and not in C^{α}, $\alpha > 0.01$, thus we expected to obtain slow convergence in C. Indeed table 4 shows obviously our expected results.
In method S7, we use method S4 as a predictor and in method S5 as a corrector.

CONCLUSIONS

We also considered several other examples such as

$$y(x) = x + \frac{3}{4}(\ell n2 - 1) + \frac{x}{4} \int_0^x \frac{y(t)y(x)}{x+t} dt \tag{3.4}$$

TABLE 4. Errors for equation (3.3) at $x = 1$. $|y(1) - y(1)|$

Method	h = .1	h = .01	h = .0025
S1	$.26169 \times 10^{-1}$	$.27253 \times 10^{-2}$	$.28269 \times 10^{-2}$
S2	$.20831 \times 10^{-1}$	$.26134 \times 10^{-2}$	$.26011 \times 10^{-2}$
S3	$.23931 \times 10^{-1}$	$.18968 \times 10^{-2}$	$.17055 \times 10^{-2}$
S4	$.36377 \times 10^{-1}$	$.19114 \times 10^{-2}$	$.18119 \times 10^{-2}$
S5	$.20031 \times 10^{-1}$	$.18161 \times 10^{-2}$	$.18060 \times 10^{-2}$
S6	$.15461 \times 10^{-2}$	$.14551 \times 10^{-2}$	$.11030 \times 10^{-3}$
S7	$.15986 \times 10^{-2}$	$.14643 \times 10^{-2}$	$.11041 \times 10^{-3}$

$$y(x) = x^{0.01} - 25x^{2.02}/101 + .25 \int_0^x y(t)y(x)dt \tag{3.5}$$

$$y(x) = \sqrt{x} - \frac{1}{3} x^2 + \frac{1}{4} \int_0^x \frac{y(t)^2 y(x)}{\sqrt{x-t}} dt, \tag{3.6}$$
$$0 \leqslant x \leqslant 1.$$

In view of the results of (3.1), (3.2) and these examples and others, we recommend using method S6 which is simple to use and is suitable for both singular and nonsingular cases. One can guess that this method is better than other methods since we use the same method through $0 \leqslant x \leqslant 1$ and always with three points. On the other hand, we recommend that method S1 be avoided. For our examples this method yielded slow convergence and some instability. It is clear that method R1 is inapplicable in the case of (3.6). Moreover, we obtained that all methods yielded solutions which converged slowly in the case of (3.6). It seems that the slow convergence is due to the singularities in (3.6).

APPENDIX A

In this appendix, we prove lemmas 2.1, 2.2 and theorem 2.2.

Proof of lemma 2.1

Suppose that the conclusion of the lemma is not true. Then there exists a sequence $\{x_n\}$, $\{\alpha_n\}$, $\{\delta_n\}$, and an $\epsilon > 0$ such that $x_n \to x$, $x \in [0, a]$, $\alpha_n \to \alpha$, $0 < \delta_n \to 0$ and

$$\int_{\alpha_n}^{\alpha_n + \delta_n} |k(t, x_n)| dt \geqslant \epsilon, \quad 0 \leqslant \alpha_n \leqslant x_n - \delta_n .$$

$$0 \leqslant \int_{\alpha_n}^{\alpha_n + \delta_n} |k(t, x_n)| dt \leqslant \int_0^a |\sigma(t, \alpha_n + \delta_n) k(t, x_n)$$
$$- \sigma(t, \alpha_n) k(t, x_n)| dt \leqslant \int_0^a |\sigma(t, x_n) k(t, x_n)$$

$$- \sigma(x,t)\,k(t,x)|\,|\sigma(t,a_n+\delta_n)-\sigma(t,a_n)|\,dt$$

$$+ \int_0^a |\sigma(t,x)\,k(t,x)|\,|\sigma(t,a_n+\delta_n)-\sigma(t,a_n)|dt$$

$$\leqslant \int_0^a |\sigma(t,x_n)\,k(x_n,t)-\sigma(t,x)\,k(t,x)|dt$$

$$+ \int_0^a |\sigma(t,x)\,k(t,x)|\,|\sigma(t,\sigma_n+\delta_n)-\sigma(t,a_n)|dt.$$

The first term on the right hand side tends to zero by A2 and since $|\sigma(t,a_n+\delta_n)-\sigma(t,a_n)| \to 0$, a.e., it follows from the Lesbeque dominated convergence theorem that the second integral also tends to zero. We have a contradiction and the lemma is proved.

Proof of lemma 2.2

Let $\epsilon > 0$. By lemma 2.1 there is an integer m such that if $h = a/m$, $p_i = (i-1)h$ and $p_{i+1} = ih$,

$$\int_{p_i}^{a_i(x)} |k(t,x)|dt < \frac{\epsilon}{4}, \qquad x \geqslant a_i(x)$$

$$a_i(x) = \begin{cases} p_i & x \leqslant p_i \\ x & p_i \leqslant x \leqslant p_{i+1} \\ p_{i+1} & x \geqslant p_{i+1} \end{cases}$$

Let

$$\phi_i(t) = \begin{cases} 1 & p_i \leqslant x \leqslant p_{i+1} \\ 0 & x \in [p_i, p_{i+1}]. \end{cases}$$

As $\phi_i \in R[0,a]$ we have from the results of [11] that for

$$v = \pm \lim_{n\to\infty} ||\Sigma\, w^{\nu}_{j,n}(x) - \int_0^x k^{\nu}(t,x)\,\phi_i(t)dt||$$

$$= \lim_{n\to\infty} ||\sum_{j\in[p_i,p_{i+1}]} w_j^{\nu}(x) \int_{p_i}^{a_i(x)} k^{\nu}(t,x)dt|| = 0.$$

Hence for sufficiently large n as $k^{\nu}(t,x) \leqslant |k(t.x)|$ we have

$$\sum_{j\in[p_i,p_{i+1}]} w_j^{\nu}(x) < \frac{\epsilon}{2}, \qquad x \in [0,a].$$

Therefore, by equation (2.3)

$$\sum_{j\in[p_i,p_{i+1}]_n} |w_j(x)| < \epsilon, \qquad x \in [0,a]$$

for all sufficiently large N. As there are only finitely many i's there exists an N such that the results of our lemma is obtained.

Proof of theorem 2.2

Let

$\epsilon_i = s(t_i) - s_n(t_i)$ or

$$\epsilon_i = \int_0^{t_i} k(t,t_i)h[t,t_i,s(t),s(t_i)]dt$$

$$- \sum_{j=1}^{i} w_j(t_i)h[t_j,t_i,s_n(t_j),s_n(t_i)]\,.$$

Therefore, from A4 we obtain

$$|\epsilon_i| \leqslant \sum_{j=0}^{i} |w_j(t_i)|\,|\epsilon_j|\,L_1 + \sum_{j=1}^{i} |w_j(t_i)|\,|\epsilon_i|\,L_2 + Q_i \quad (4.1)$$

where

$$Q_i \equiv |\int_0^{t_i} k(t,t_i)h[t,t_i,s(t),s(t_i)]dt$$

$$- \sum_{j=1}^{i} w_j(t_i)h[t_j,t_i,s(t_j),s(t_i)]|\,. \quad (4.2)$$

From corollary 2.2 for N sufficiently large $L_1|w_i(x)| < \frac{\epsilon}{2}$ for each $1 \leqslant i \leqslant n$, $n > N$ and $x \in [0,a]$. Therefore, in view of corollary 2.3, one can show that for N sufficiently large

$$\max_{1\leqslant i\leqslant n} (1 - L_1|w_i(x_i)| - L_2 \sum_{j=0}^{i} |w_j(x_i)|)^{-1} < \frac{2}{\beta}. \quad (4.3)$$

Hence, there exist positive constants

$$L' = \max_{1\leqslant i\leqslant n} \left[\frac{L_1}{1 - |w_i(x_i)|L_1 - \sum_{j=1}^{i} |w_j(x_i)|L_2} \right] \quad (4.4)$$

$$Q' = \max_{1\leqslant i\leqslant n} \frac{Q_i}{(1 - |w_i(x_i)|L_1 - \sum_{j=1}^{i} |w_j(x_i)|L_2)} \cdot (4.5)$$

Using inequality (4.1) and equations (4.4), (4.5) we get

$$|\epsilon_i| \leqslant L' \sum_{j=0}^{i-1} w_j(x_i)||\epsilon_j| + Q'. \quad (4.6)$$

By using lemma 2.2, we obtain that for N sufficiently large there exist points $p_1, \ldots, p_R$ so that

$$\max_{1\leqslant i\leqslant n} \sum_{j\in[p_i,p_{i+1}]_n} |w_j(x_i)| < \frac{\beta}{L'}.$$

Now we use lemma 1 in [7] to obtain that for $x_i \in [p_1, p_2]$, $|\epsilon_i| \leqslant 2Q$ (we take $\beta = \frac{1}{2}$ and $\eta = 0$, in lemma 1 in [7]). We also obtained that for x_i in the r-th interval, $1 \leqslant r \leqslant R$,

$$|\epsilon_i| \leqslant \frac{1}{2} \sum_{i=1}^{r} 2^i Q' < (2^r - 1)Q' \leqslant (2^R - 1)Q'.$$

Hence, we obtain that $\max_{1\leqslant i\leqslant n} |\epsilon_i| \to 0$ as $n \to \infty$.

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